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High-Order Discontinuous Galerkin For Numerical Simulation of Elastic Wave Propagation

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Abstract — We present a study of elastic wave propagation in isotropic media. The Discontinuous Galerkin Method is applied to solve the elastodynamic equations which represent elastic wave propagation. The elastodynamic equations are transformed into a stress-velocity formulation. The Discontinuous Galerkin Method is a finite element that allows a discontinuity of the numerical solution at element interface. Through a proper choice of the flux computation points, the method only requires communication between elements that have common faces. The utilization of high-order Legendre polynomials as basis functions has been shown to be more efficient in reducing the numerical dispersion and numerical dissipation. Discontinuous Galerkin Method is a compact method, high-order basis functions can be used easily without any essentially difficulty and even spectral accuracy becomes obtainable. Temporal discretization utilized explicit staggered leapfrog method. We compare the numerical results to the exact solutions and the comparison shows a good agreement.

Keywords — Elastic wave propagation, Discontinuous Galerkin, high order basis functions.

1. Introduction

The discontinuous galerkin (DG) method originally developed by Reed and Hill (1973) for the solution of neutron transport problem. Le Saint and Raviart (1974) were the first to put the method on a firm mathematical base. The technique lay dormant for several years before becoming popular. Cockburn and Shu [1,2] developed DG method to solve convection-diffusion problem and extended it to multidimensional systems case. Recently, the DG formulation of the finite element method has been increasingly use in computational fluid dynamics (CFD) [3,4], computational electromagnetic (CEM) [5] and computational aeroacoustics (CAA) [6,7].

It is well known that highly accurate methods are required for long time simulations of wave propagation, which are

essentially non dispersive and non dissipation [8]. Many numerical schemes such as finite difference and finite element are developed to study wave propagation. Among the schemes, the DG method provides an attractive approach to solve problems containing discontinuities, such as those arise in hyperbolic systems. The DG method allows more general mesh configuration and interelement continuity is not required. The basis function is discontinuous across mesh boundaries. Through a proper choices of flux computation points, the method only requires communication between mesh that have common faces. No global matrix inversion is required and the problem can be solved locally in each mesh. The DG method can be regarded as cross between a finite volume and finite element method and it has many of the good properties of both.

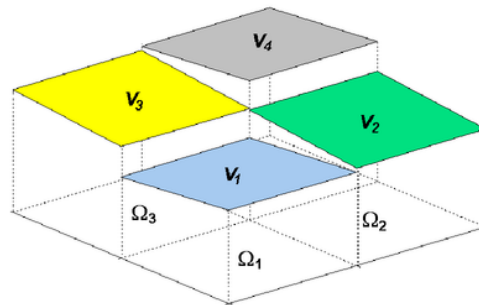


Figure 1. Mesh and solution of Discontinuous Galerkin Method.

Lowrie considered the space-time DG method which involves discontinuous mesh in both space and time to solve hyperbolic laws [9]. The method requires excessive resources to be useful for practical application. Atkins and Shu described a quadrature free formulaton for DG method [6]. They apply the formulation to linearized Euler equations.

Stanescu *et al.* combined DG method with spectral element method to solve aircraft engine noise scattering [10]. They used high-order Legendre polynomials as basis functions. The numerical results show that trends of the noise field are well predicted. Li developed adaptive space-time DG method to solve elastodynamic equations [11]. Triangular mesh and linear basis function were used. The numerical results have a good agreement with exact solution.

In this paper, elastodynamic equations, which described elastic wave propagation, are solved using DG method. High order Legendre polynomials are used as basis functions. The elastodynamic equations will be discretized using rectangular mesh and explicit Leapfrog method is used as time integration method.

2. Governing Equations

The Elastodynamic Equations are set of linear hyperbolic equations. The equations are transformed into the following first-order hyperbolic system via stress-velocity formulation.

$$\frac{\partial v_x}{\partial t} = \frac{1}{\rho} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) + f_x \quad (1.a)$$

$$\frac{\partial v_y}{\partial t} = \frac{1}{\rho} \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right) + f_y \quad (1.b)$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_y}{\partial y} \quad (1.c)$$

$$\frac{\partial \tau_{yy}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_y}{\partial y} + \lambda \frac{\partial v_x}{\partial x} \quad (1.d)$$

$$\frac{\partial \tau_{xz}}{\partial t} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \quad (1.e)$$

In which (v_x, v_y) is the velocity vector, $(\tau_{xx}, \tau_{yy}, \tau_{xy})$ is the stress tensor, (f_x, f_y) is body force vector, ρ is the density, λ and μ are Lamé coefficient. Stress-velocity formulation for one-dimensional propagation in the x direction are:

$$\frac{\partial v_y}{\partial t} - \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial x} = 0 \quad (2.a)$$

$$\frac{\partial \tau_{xy}}{\partial t} - \mu \frac{\partial v_y}{\partial x} = 0 \quad (2.b)$$

3. DG Discretization

A. One-Dimensional Discretization

We present one-dimensional DG Discretization in one dimension to begin with, and extend them to two dimensions. We divide the interval of domain into subintervals $\mathcal{Q}_j = [x_j, x_{j+1}]$, $j = 0, \dots, N$. In each subinterval (element), we expand the velocity and stress in terms of Legendre cardinal functions:

$$\begin{aligned} v_y &= \sum_{i=0}^m h_i(x) (v_y)_i = [h] [v_y] \\ \tau_{xy} &= \sum_{i=0}^m h_i(x) (\tau_{xy})_i = [h] [\tau_{xy}] \end{aligned} \quad (3)$$

The Legendre cardinal functions are written as below:

$$h_i(x) = \frac{(x^2 - 1) L'_m(x)}{m(m+1) L_m(x_i) (x - x_i)} \quad (4)$$

L_m are m -th order Legendre Polynomial.

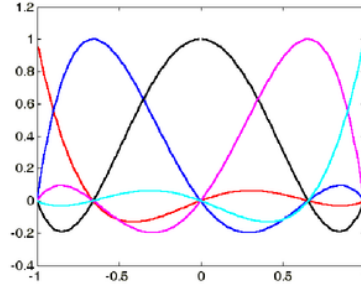


Figure 2. Fourth-order Legendre cardinal function.

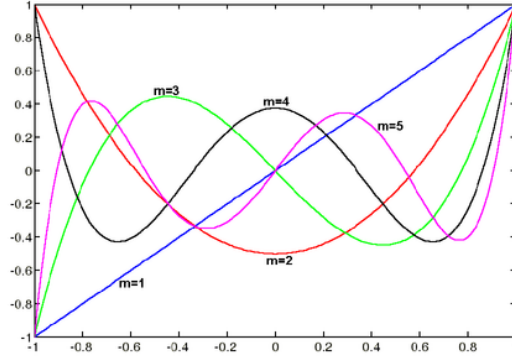


Figure 3. Legendre Polynomials.

As example, we sample equation (2.a) according Galerkin's procedure using $[h]$ as testing function:

$$\int_{x_j}^{x_{j+1}} [h]^T [h] \frac{\partial v_y}{\partial t} dx - \frac{1}{\rho} \int_{x_j}^{x_{j+1}} [h]^T \frac{\partial [h]}{\partial x} [\tau_{xy}] dx = 0 \quad (5)$$

Integrate by parts the term with spatial derivative:

$$\int_{x_j}^{x_{j+1}} [h]^T [h] \frac{\partial v_y}{\partial t} dx - \frac{1}{\rho} \left(- \int_{x_j}^{x_{j+1}} \frac{\partial [h]^T}{\partial x} [h] [\tau_{xy}] dx + [h]^T [h] [\tau_{xy}]_{x_j}^{x_{j+1}} \right) = 0 \quad (6)$$

We make a modification to approximate $[\tau_{xy}]_{x_j}^{x_{j+1}}$ by $[\hat{\tau}_{xy}]_{x_j}^{x_{j+1}}$,

integrate by parts the second term:

$$\int_{x_j}^{x_{j+1}} [h]^T [h] \frac{\partial [v_y]}{\partial t} dx - \frac{1}{\rho} \left(\int_{x_j}^{x_{j+1}} \frac{\partial [h]^T}{\partial x} [h] [\tau_{xy}] dx + [h]^T [h] [\hat{\tau}_{xy} - \tau_{xy}]_{x_j}^{x_{j+1}} \right) = 0 \quad (7)$$

The numerical flux $(\hat{\tau}_{xy})$ at the interface between elements is approximated by using average flux:

$$\hat{\tau}_{xy} = \frac{1}{2} (\tau_{xy}^+ + \tau_{xy}^-) \quad (8)$$

The (+) notation implies the limit of τ_{xy} from outside of the element and the (-) notation implies the limit of τ_{xy} from inside of the element.

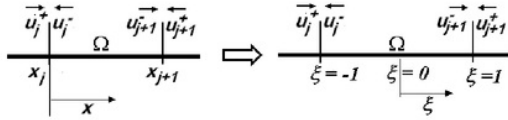


Figure 4. Flux and local coordinate.

The global coordinate (x) are mapped onto local coordinate (ξ), each element $\Omega = [x_j, x_{j+1}]$ is then mapped onto master element $\Omega = [-1, 1]$. All integral are evaluated numerically using Gauss Lobatto Legendre (GLL) quadrature. GLL quadrature has accuracy of order $(2m-2)$.

The final expressions for the DG discretizations of equation (2.a) is:

$$\mathbf{M} \frac{\partial [v_y]}{\partial t} - \frac{1}{\rho} \mathbf{C} [\tau_{xy}] - \frac{1}{\rho} \left([h]^T [h] [\hat{\tau}_{xy} - \tau_{xy}]_{x_j}^{x_{j+1}} \right) = 0 \quad (9)$$

where: $\mathbf{M}$ is mass matrix and $\mathbf{C}$ is advection matrix.

$$\mathbf{M} = \mathbf{J} \mathbf{W} ; \mathbf{J} = \frac{dx}{d\xi} ; \mathbf{C} = \mathbf{W} \mathbf{D} \quad (10.a)$$

$$\mathbf{D} = \frac{\partial h_j(\xi_i)}{\partial \xi} \begin{cases} L_m(\xi_i) & \text{if } i \neq j \\ 0 & \text{if } i = j, i \neq 0, m \\ -\frac{m(m+1)}{4} & \text{if } i = j = 0 \\ \frac{m(m+1)}{4} & \text{if } i = j = m \end{cases} \quad (10.b)$$

$$\mathbf{W} = \begin{bmatrix} \omega_1 & 0 & - & 0 \\ 0 & \omega_2 & - & 0 \\ - & - & - & - \\ 0 & 0 & - & \omega_m \end{bmatrix} \quad (10.c)$$

$\mathbf{J}$ is the Jacobian $\mathbf{W}$ is the weight matrix and ω_j are the weights and $\mathbf{D}$ is the differential matrix.

B. Two-Dimensional Discretization

The domain is divided into non-overlapping rectangular elements within which a high order polynomial expansion is used. We mapped the global coordinates (x, y) onto local coordinates (ξ, η). The basis (trial or test function) is constructed by taking a product of the one-dimensional basis which can be thought of as one-dimensional tensors. We take expansion of v_y as example:

$$v_y = \sum_{k=1}^m \sum_{l=1}^m v_{y,kl} h_k(\xi) h_l(\eta) \quad (11)$$

Figure 5 shows the two-dimensional mesh and figure 6 shows the basis functions. The DG discretization of 2-dimensional case is analogous to the one-dimensional case, for more detail see Kamiadakis and Sherwin [3].

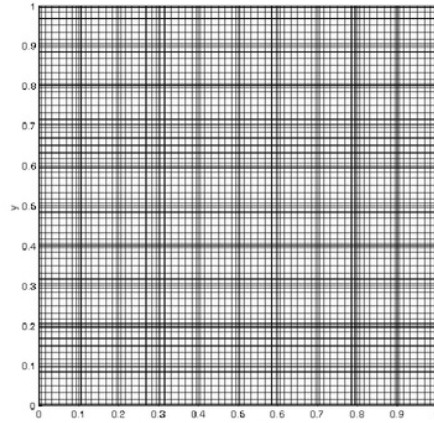


Figure 5. Two-dimensional mesh.

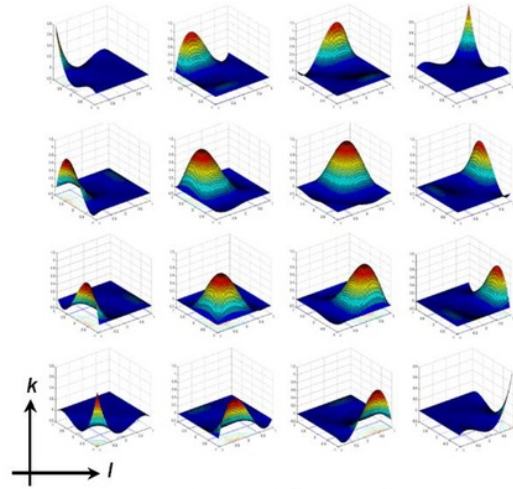


Figure 5. Two-dimensional 3rd order basis functions.

4. Numerical Results and Discussion

A. One-dimensional case

To evaluate the accuracy of DG scheme, the following initial conditions are taken to perform numerical simulations:

$$\begin{aligned} v_y(x, 0) &= 0 \\ \tau_{xy}\left(x, \frac{\Delta t}{2}\right) &= \exp(-\ln 2x^2/9) \quad ; -100 \leq x \leq 100 \end{aligned} \quad (12)$$

The results are compared to known exact solution, the exact solution in this case is:

$$\tau_{xy}(x, t) = \left(\frac{\exp(-\ln 2((x-t)^2/9)) + \exp(-\ln 2((x+t)^2/9))}{2} \right) \quad (13)$$

We take $\Delta t = 0.05$ and fixed number of elements = 50. Figure 5 shows the mean absolute error (MAE) versus polynomial order. The MAE is plotted on logarithmic scale and we can deduce exponential (spectral) convergence from approximately straight line on the plot.

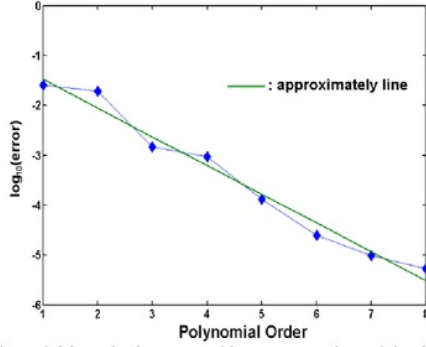


Figure 6. Mean absolute error with respect to polynomial order.

We take $\Delta t = 0.05$ and fixed polynomial order = 5. As the number of elements increases, the order of accuracy also increases linearly. Figure 8a – 8c show the propagation of the shear wave.

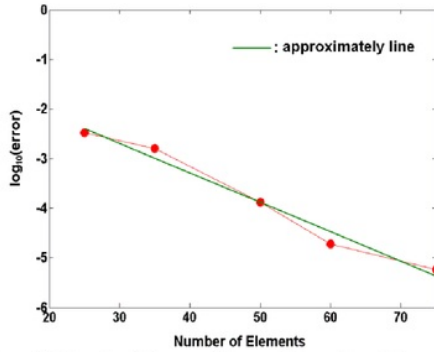


Figure 7. Mean absolute error with respect to number of elements.

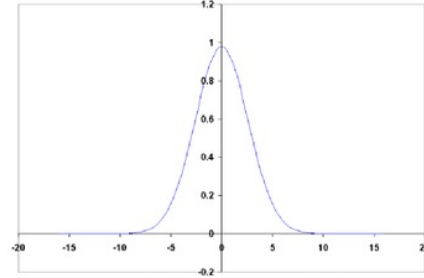


Figure 8a. Profile of τ_{xy} at $t = 0.525$.

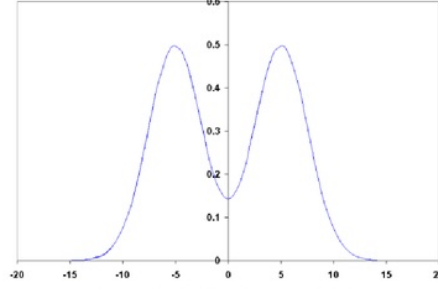


Figure 8b. Profile of τ_{xy} at $t = 5.525$.

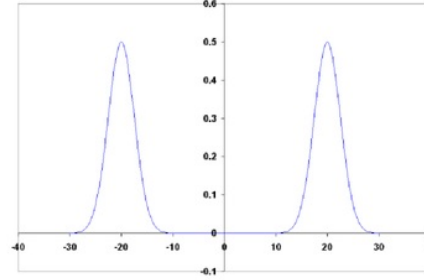


Figure 8c. Profile of τ_{xy} at $t = 20.025$.

B. Two-dimensional case

We consider a problem defined on the unit square $(0,1) \times (0,1)$ with $(v_x, v_y) = 0$ on the boundary. We choose the body force as [11]:

$$\begin{aligned} f_x &= (\lambda + \mu)(1-2x)(1-2y)\sin \omega t \\ f_y &= (\rho \omega^2 xy(1-x)(1-y) - 2\mu y(1-y) \\ &\quad - 2(\lambda + 2\mu)x(1-x))\sin \omega t \end{aligned} \quad (14)$$

The exact solutions are:

$$\begin{aligned} v_x &= 0 \\ v_y &= -xy(1-x)(1-y)\cos \omega t \\ \tau_{xx} &= -\lambda x(1-x)(1-2y)\sin \omega t \\ \tau_{yy} &= -(\lambda + 2\mu)x(1-x)(1-2y)\sin \omega t \\ \tau_{xy} &= -\mu y(1-y)(1-2x)\sin \omega t \end{aligned} \quad (15)$$

Plain strain condition is assumed and the parameters are taken as $E = 1.0$, $\nu = 0.3$, $\rho = 1.0$, $\omega = 1.0$, time step = 0.002, numbers of element = 100 and polynomial order = 8. The pulsation of v_y with time is given in Figure 9a – 9e and the pulsation of τ_{xy} is given in Figure 10a – 10e.

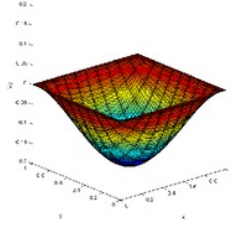


Figure 9a. v_y field at $t = 0$.

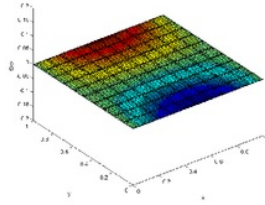


Figure 10a. τ_{xy} field at $t = 0$.

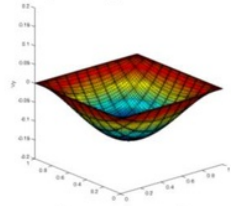


Figure 9b. v_y field at $t = 0.25$.

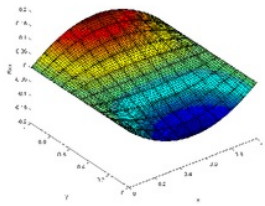


Figure 10b. τ_{xy} field at $t = 0.25$.

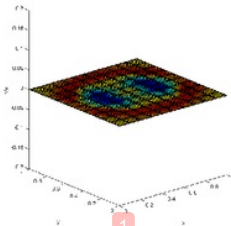


Figure 9c. v_y field at $t = 0.5$.

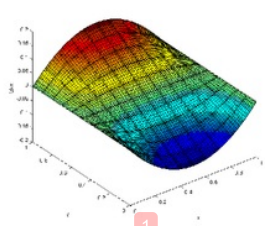


Figure 10c. τ_{xy} field at $t = 0.5$.

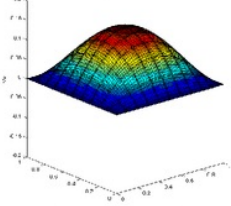


Figure 9d. v_y field at $t = 0.75$.

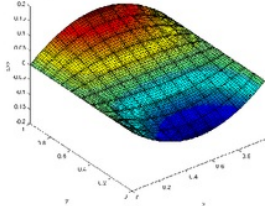


Figure 10d. τ_{xy} field at $t = 0.75$.

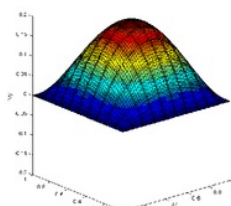


Figure 9e. v_y field at $t = 1.0$

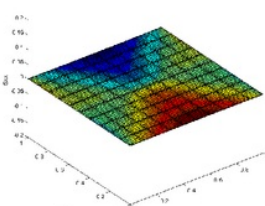


Figure 10e. τ_{xy} field at $t = 1.0$

Comparisons with the exact solutions are shown in figure 11a and 11b for v_y and τ_{xy} profile, very good agreements are found. Figure 12 shows that trends of DG errors almost constant in time, the growth of error (dispersive & dissipation error) in DG method can be reduced by using high-order basis.

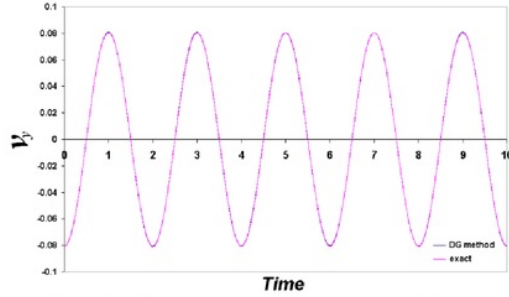


Figure 11a. Histories of v_y responses at the point (0.4,0.4).

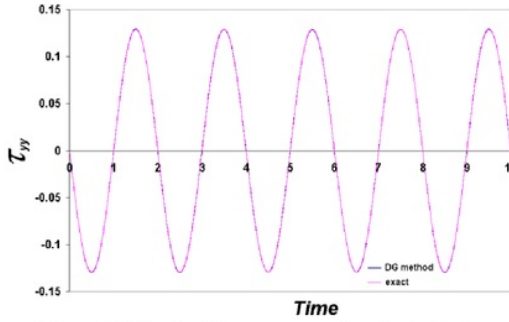


Figure 11b. Histories of τ_{xy} responses at the point (0.4,0.4).

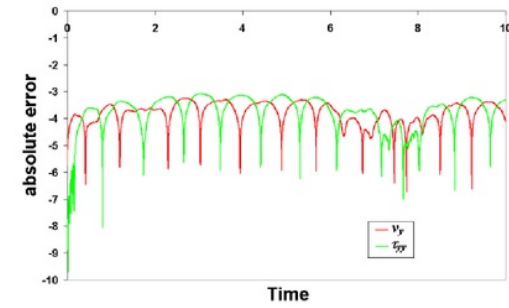


Figure 12. Histories of absolute errors of v_y and τ_{xy} at the point (0.4,0.4).

2. Conclusions

We have derived and applied Discontinuous Galerkin methods in the numerical modeling of elastic wave propagation and numerical results have a good agreement with exact solution. In simple domain, the exponential

convergence can be achieved by either increasing the number of elements, called *h-refinement*, or by increasing the polynomial order of a fixed number of elements, called *p-refinement*.

For future research, we plan to extend the DG method for solving problems with irregular domain and apply *hp adaptive* technique to increase the accuracy and to reduce computational costs.

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